# Accelerator dynamics of a fractional kicked rotor

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It is shown that the Weyl fractional derivative can quantize an open system. A fractional kicked rotor is studied in the framework of the fractional Schrödinger equation. The system is described by the non-Hermitian Hamiltonian by virtue of the Weyl fractional derivative. Violation of space symmetry leads to acceleration of the orbital momentum. Quantum localization saturates this acceleration, such that the average value of the orbital momentum can be a direct current and the system behaves like a ratchet. The classical counterpart is a nonlinear kicked rotor with absorbing boundary conditions.

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Application of fractional calculus to quantum processes is a new approach to the study of fractional properties of quantum phenomena [1–6]. In this Brief Report we consider quantum chaotic dynamics of a fractional kicked rotor (FKR). The Hamiltonian of the system is

$$\hat{H} = \hat{\mathcal{T}} + \epsilon \cos x \sum_{n=-\infty}^{\infty} \delta(t-n), \tag{1}$$

where $\epsilon$ is the amplitude of the periodic perturbation which is a train of $\delta$ kicks. The kinetic part of the Hamiltonian is modeled by the fractional Weyl derivative

$$\hat{\mathcal{T}} = (-i\tilde{h})^{\alpha}\mathcal{W}^{\alpha}/\alpha, \tag{2}$$

where $\tilde{h}$ is the dimensionless Planck constant, and $\alpha=2-\beta$ with $0<\beta<1$. When $\alpha=2$ Eq. (1) is the quantum kicked rotor [7]. For a periodic function $f(x)=\Sigma \bar{f}_k e^{-ikx}$, the Fourier transform property determines the fractional Weyl derivative $\mathcal{W}^{\alpha}$ in the following simplest way (see [3], Chap. 4.3):

$$\mathcal{W}^{\alpha} f(x) = \sum_{n=-\infty}^{\infty} (-ik)^{\alpha}\bar{f}_k e^{-ikx}. \tag{3}$$

Since only periodic functions are considered here, this oversimplified definition is sufficient without the burden of fractional calculus details [8]. Thus, the kinetic term in the Hamiltonian (1) is defined on the basis $|k\rangle = e^{ikx}/\sqrt{2\pi}$ as follows:

$$\hat{\mathcal{T}}|k\rangle = \mathcal{T}(k)|k\rangle = \frac{(\tilde{h}k)^{2-\beta}}{2-\beta}|k\rangle. \tag{4}$$

This non-Hermitian operator has complex eigenvalues for $k<0$, which are defined on the complex plane with a cut from 0 to $-\infty$, such that $1^{-\beta}=1$ and $(-1)^{-\beta} = \cos\beta\pi - i\sin\beta\pi$, and therefore $k^{-\beta}=|k|^{-\beta}e^{-i\pi\beta(k)}$, where $\beta(k)=\beta[1-\mathrm{sgn}(k)]/2$ [9]. It is worth mentioning that the fractional derivative in Eq. (2) appears naturally in quantum lattice dynamics with long-range interaction [4], where $(-ik)^{\alpha}$ is a particular case of a polylogarithm (see Appendix in Ref. [4]).

A quantum map for the wave function $\psi(x,t)$ is

$$\psi(x,t+1) = \hat{U}\psi(x,t), \tag{5}$$

where the evolution operator on the period

$$\hat{U} = \exp\left(\frac{-i\epsilon\cos x}{\tilde{h}}\right)\exp\left(\frac{-i\hat{\mathcal{T}}}{\tilde{h}}\right) \tag{6}$$

describes free dissipative motion and then a kick. The dynamics of the FKR is studied numerically, where Eq. (4) enables one to use the fast Fourier transform as an efficient way to iterate the quantum map (5). A specific property of this Hamiltonian dynamics is quantum dissipation resulting in probability leakage and described by the survival probability

$$P(t) = \langle\psi(t)|\psi(t)\rangle = \sum_{n=-\infty}^{\infty} |f_n|^2, \tag{7}$$

where $|f_n|^2$ is the probability of level occupation at time $t$. The initial occupation is $f_n(t=0)=\delta_{n,0}$. Another specific characteristic is the nonzero mean value of the orbital momentum,

$$\langle p(t)\rangle = \tilde{h}\frac{\sum_n n|f_n(t)|^2}{\sum_n |f_n(t)|^2}, \tag{8}$$

due to the asymmetry of the quantum kinetic term $\hat{\mathcal{T}}$. Results of the numerical study of the quantum map (5) are shown in Figs. 1–4. The quantum dissipation leads to an asymmetrical distribution of the level occupation $|f_n(t)|$ (see Fig. 1) that results in a nonzero first moment of the orbital momentum $\langle p\rangle \sim t^{\gamma_1}$ in Fig. 2. Quantum localization saturates the acceleration with time. This accelerator dynamics is accompanied by the power law decay of the survival probability $P(t)\sim t^{-\gamma_2}$ with the exponent $\gamma_2\approx 0.71$ shown in Fig. 3, and then the decay rate increases with time due to quantum effects. Quantum localization affects strongly both $\gamma_1$ and $\gamma_2$. By increase of the quantum parameter, when $\tilde{h}=0.76$, the exponent $\gamma_1$ approaches zero (in Fig. 4 the slope is $10^{-5}$), and the survival probability decays at the rate $\gamma_2\approx 0.99$.

To understand the obtained numerical results and the physical relevance of the fractional Schrödinger equation,

 

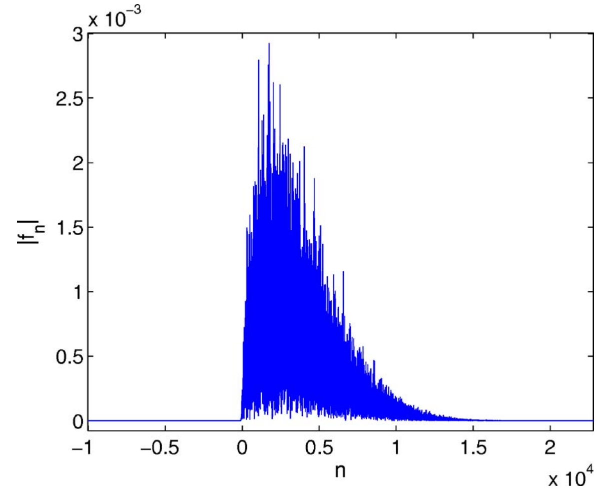


FIG. 1. (Color online) Level occupation distribution (after 2000 iterations) for $\epsilon=3$, $\beta=0.01$, $\tilde{h}=0.02$.

the classical limit $\tilde{h}\to 0$ is performed in the Wigner representation. Thus, the system is described by the Wigner function $W(x,p,t)$ which is a $c$-number projection of the density matrix in the Weyl rule of association between $c$ numbers and operators. The Weyl transformation of an arbitrary operator function $G(\hat{x},\hat{p})$ is [10,11]

$$F(x,p)=\mathrm{Tr}[G(\hat{x},\hat{p})\Delta(x-\hat{x},p-\hat{p})], \tag{9}$$

where $F(x,p)$ is a $c$-number function and $\Delta(x-\hat{x},p-\hat{p})$ is a projection operator which acts as a two-dimensional Fourier transform. For the cylindrical phase space the projection operator is [12]

$$\Delta(x-\hat{x},p-\hat{p})=\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}d\xi\, e^{im(x-\hat{x})+i\xi(p-\hat{p})}. \tag{10}$$

This operator determines the inverse transform as well:

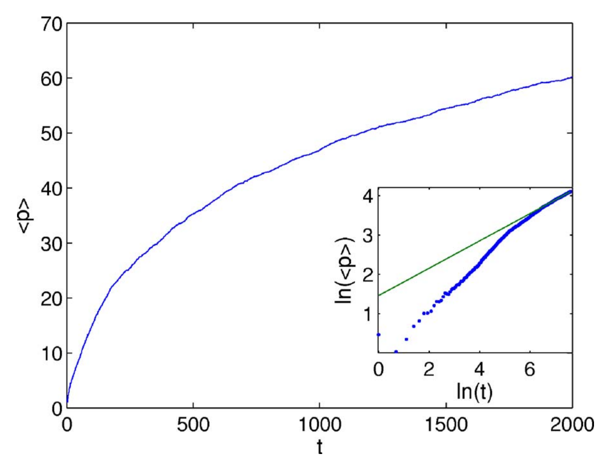


FIG. 2. (Color online) Acceleration of the average orbital momentum for the same parameters as in Fig. 1. The inset is a log-log plot, and the solid line corresponds to $\gamma_1=0.35$ obtained by a least-squares calculation.

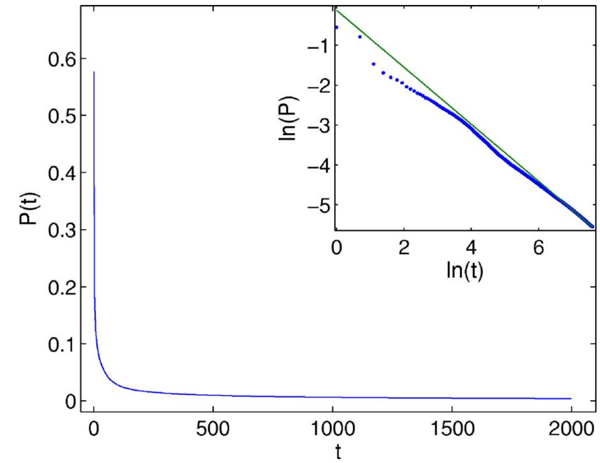


FIG. 3. (Color online) Decay of the survival probability $P(t)$. The inset is a log-log plot, and the solid line corresponds to $\gamma_2=0.71$ obtained by a least-squares calculation.

$$G(\hat{x},\hat{p})=\sum_{k=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}F(x,\tilde{h}k)\Delta(x-\hat{x},\tilde{h}k-\hat{p}), \tag{11}$$

where $p=\tilde{h}k$. The quantum map for the density matrix $\hat{\rho}(t)$ is

$$\hat{\rho}(t+1)=\hat{U}^{\dagger}\hat{\rho}(t)\hat{U}. \tag{12}$$

Therefore, evolution of the Wigner function

$$W(t,x,p)=\mathrm{Tr}[\hat{\rho}(t)\Delta(x-\hat{x},p-\hat{p})]$$

for the period determined by the map (12) is

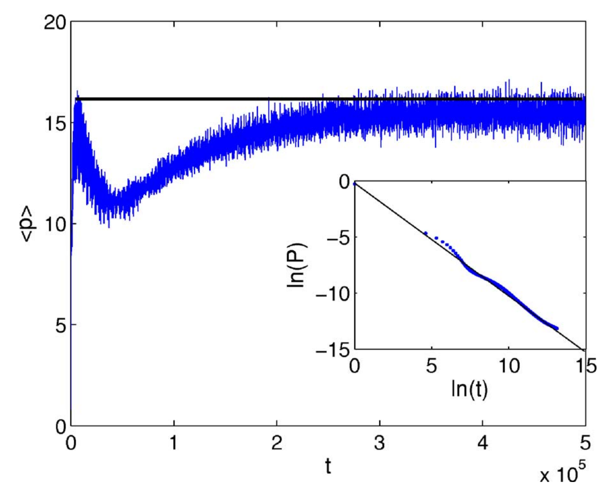


FIG. 4. (Color online) Quantum saturation of $\langle p(t)\rangle$ due to localization when $\tilde{h}=0.76$, $\beta=0.05$, and with the same $\epsilon$ as in Figs. 1–3. The slope of the solid line is $10^{-5}$. The inset shows the power law decay of $P(t)\sim t^{-\gamma_2}$ with $\gamma_2\approx 0.99$ due to the linear interpolation.

$$W(t+1,x,p) = \mathrm{Tr}[\hat{U}^\dagger \hat{\rho}(t)\hat{U}\Delta(x-\hat{x},p-\hat{p})] = \sum_{k'=-\infty}^{\infty}\int_0^{2\pi} K_{\tilde{h}}(x,p|x',p')W(t,x',p')dx', \tag{13}$$

where $K_{\tilde{h}}(x,p|x',p')$ is the Green's function for the period,

$$K_{\tilde{h}}(x,p|x',p') = \sum_m \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{im(x-x'+\xi')}e^{i\xi'(p'-p)} \times \exp\left(\frac{i}{\tilde{h}}\mathcal{T}^*(p+\tilde{h}m/2) - \frac{i}{\tilde{h}}\mathcal{T}(p-\tilde{h}m/2)\right) \times \exp\left(\frac{i\epsilon}{\tilde{h}}\cos(x'+\tilde{h}\xi'/2) - \frac{i\epsilon}{\tilde{h}}\cos(x'-\tilde{h}\xi'/2)\right). \tag{14}$$

The trace is $\mathrm{Tr}[\cdots]=\Sigma_k\langle k|\cdots|k\rangle$. In the classical limit $\tilde{h}\to 0$, we obtain in Eq. (14) that the difference of the perturbations in the exponential is $-i\epsilon\cos x$, while the difference of the kinetic terms is $imp^{1-\beta}\equiv im\omega(p)$ for $p>0$ and $-2\sin(\beta\pi)\mathcal{T}(|p|)/\tilde{h}$ for $p<0$. The last term diverges at $\tilde{h}=0$ and yields identically zero for the Green's function $K_{\tilde{h}=0}(p<0)\equiv 0$. Thus, the classical Green's function

$$K_{\tilde{h}=0}(x,p|x'p') = \Theta(p)\delta(x-x'-\omega(p))\delta(p-p'-\epsilon\sin x') \tag{15}$$

corresponds to the classical map $\mathcal{M}$,

$$p_{n+1} = p_n + \epsilon\sin x_n, \quad x_{n+1} = x_n + \omega(p_{n+1}), \tag{16}$$

of the nonlinear kicked rotor with the nonlinear frequency $\omega(p)$, and absorbing boundary conditions for $p<0$, which the Heaviside function $\Theta(p)$ reflects.

Therefore, the fractional Hamiltonian (1) corresponds to the open system Eqs. (15) and (16). Chaotic dynamics of this open system takes place in the upper half of the cylindrical phase space. The stability property is determined by the trace of the linearized map $\partial\mathcal{M}$,

$$\mathrm{Tr}[\partial\mathcal{M}] = 2 + \epsilon(1-\beta)p^{-\beta}\cos x. \tag{17}$$

For any $\epsilon$ there are stable regions $\{\Delta x,\Delta p\}$ determined by the locus of elliptic points $\{x_e,p_e\}$ (see Fig. 5)

$$x_e = \arccos\left(-\frac{2p_e^\beta}{\epsilon(1-\beta)}\right). \tag{18}$$

The presence of this infinite regular elliptic island structure, which leads to the stickiness of chaotic trajectories [13], also results in the power law decay of the survival probability for the quantum counterpart in Fig. 3. It should be stressed that the nature of the power law decay of quantum long-time dynamics differs from the classical one. On the Ehrenfest time scale, when quantum dynamics is described approximately by the classical trajectories, the rate of the quantum probability leakage is determined by the classical exponent due to the classical stickiness phenomenon [14]. The situation changes essentially for long-time quantum dynamics; namely, the power law decay of the survival probability, which is shown in Fig. 3, is now due to the quantum tunneling between the integrable interior of the stability islands and the chaotic sea. This power law phenomenon due to quantum tunneling has been the subject of extensive studies in quantum chaos [15].

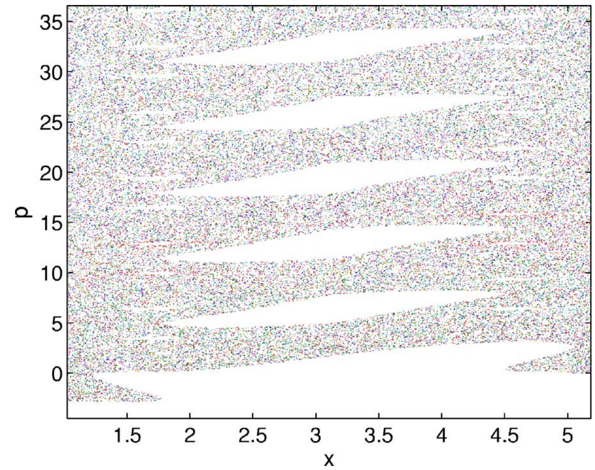


FIG. 5. (Color online) Phase portrait of the classical map without absorption, after 20 000 iterations of 15 initial conditions for $\epsilon=3$ and $\beta=0.01$.

Quantum localization leads to the exponential restriction of the initial profile spreading in the orbital momentum space from above. This property results in saturation of the acceleration of $\langle p(t)\rangle$; namely, at $t\to\infty$ it follows that $\langle p(t)\rangle \to \mathrm{const}$. Such a behavior is found for $\tilde{h}=0.76$ and $\beta=0.05$. In Fig. 4 one sees a direct current of $\langle p(t)\rangle$ for $5\times 10^5$ iterations and $K=3$. This double impact of asymmetric absorption and quantum localization leads, asymptotically, to a quantumlike ratchet which differs from the quantum one obtained on a classical chaotic attractor [16].

It is worth mentioning that, in the class of periodic functions, eigenvalues of the unperturbed Hamiltonian $\mathcal{T}$ coincide with $\hat{H}_0(\hat{p})=[(-i\tilde{h})\partial/\partial x]^\alpha$, and have the same classical limit of Eq. (15). This local derivative has a classical counterpart with the Hamiltonian $H_0(p)=p^\alpha$ which does not coincide with Eq. (15); namely, the Hamiltonian $H_0(p)$ is a classical system with dissipation for $p<0$, while the map $\mathcal{M}$ in Eqs. (15) and (16) is an open system where a particle is set apart from the dynamics for $p<0$.

The fractional Schrödinger equation

$$i\tilde{h}\partial_t\psi = (-i\tilde{h})^\alpha\mathcal{W}^\alpha\psi \tag{19}$$

describes quantum *dissipative Hamiltonian* dynamics. The classical counterpart is a nonlinear motion with dispersion $\omega(p)$ realized on the upper half plane of the phase space with absorption in the lower half plane. It has well-defined physical meaning. Therefore, the fractional Schrödinger equation (19) can be a generalized approach for any functions for which the Fourier transform is valid. In this case, the opposite classical-to-quantum transition can be performed by determining the Heaviside function in Eq. (14),

$$e^{i\omega(p)z}\Theta(p) = \lim_{\tilde{h}\to 0} \exp\left[\frac{i}{\tilde{h}}\mathcal{T}^*\left(p+\frac{\tilde{h}z}{2}\right) - \frac{i}{\tilde{h}}\mathcal{T}\left(p-\frac{\tilde{h}z}{2}\right)\right],$$

where $\mathcal{T}(p)$ is uniquely defined by the condition $\omega(p) = \mathcal{T}'(p)$. Thus, fractional derivatives quantize classical open systems in the framework of the non-Hermitian Hamiltonians. It is also worth mentioning complex scaling, which is a powerful method for the treatment of divergences of wave functions of systems with non-Hermitian Hamiltonians [17]. The application of complex scaling to the fractional Schrödinger equations is an interesting approach that will be studied in future.

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[8] Fractional derivation was developed as a generalization of integer order derivatives and is defined as the inverse operation to the fractional integral. Fractional integration of the order of $\alpha$ is defined by the operator [see, e.g., [3] and I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999)]

$${}_aI_x^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(y)(x-y)^{\alpha-1}dy,$$

where $\alpha>0$, $x>a$, and $\Gamma(z)$ is the Gamma function. Therefore, the fractional derivative is the inverse operator to ${}_aI_x^\alpha$ as ${}_aD_x^\alpha f(x) = {}_aI_x^{-\alpha}$ and ${}_aI_x^\alpha = {}_aD_x^{-\alpha}$. Its explicit form is

$${}_aD_x^{-\alpha} = \frac{1}{\Gamma(-\alpha)}\int_a^x f(y)(x-y)^{-1-\alpha}dy.$$

For arbitrary $\alpha>0$ this integral diverges, and as a result of a regularization procedure there are two alternative definitions of ${}_aD_x^{-\alpha}$. For an integer $n$ defined as $n-1<\alpha<n$, one obtains the Riemann-Liouville fractional derivative of the form

$${}_aD_{RL}^{\alpha}f(x) = (d^n/x^n)\,{}_aI_x^{n-\alpha}f(x),$$

and fractional derivative in the Caputo form

$${}_aD_C^{\alpha}f(x) = {}_aI_x^{n-\alpha}f^{(n)}(x).$$

There is no constraint on the lower limit $a$. For example, when $a=0$, one has

$${}_0D_{RL}^{\alpha}x^{\beta} = \frac{x^{\beta-\alpha\Gamma(\beta+1)}}{\Gamma(\beta+1-\alpha)}$$

$${}_aD_C^{\alpha}f(x) = {}_0D_{RL}^{\alpha}f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)},$$

and ${}_aD_C^\alpha[1]=0$, while ${}_0D_{RL}^\alpha[1]=x^{-\alpha}/\Gamma(1-\alpha)$. When $a=-\infty$, the resulting Weyl derivative is

$$\mathcal{W}^\alpha \equiv {}_{-\infty}D_W^\alpha = {}_{-\infty}D_{RL}^\alpha = {}_{-\infty}D_C^\alpha.$$

One also has ${}_{-\infty}D_W^\alpha e^x = e^x$. This property is convenient for the Fourier transform $\mathcal{F}[\mathcal{W}^\alpha f(x)] = (ik)^\alpha \bar{f}(k)$, where $\mathcal{F}[f(x)] = \bar{f}(k)$.

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